Interpolating geometries, fivebranes and the Klebanov-Strassler theory

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Based on:
[Maldacena,DM] JHEP 1001:104,2010,
[Gaillard,DM,Núñez,Papadimitriou] to appear

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Plan of the seminar

1. Supersymmetric geometries
2. Interpolating supersymmetric geometries
3. The unwarped resolved deformed conifold
4. The baryonic branch from fivebranes
5. Fivebranes from the baryonic branch
6. Adding flavours
7. A $G_2$ story (if there is time - probably there isn’t)
8. Outlook
Motivations

• Systematic studies of **supersymmetric geometries** of String/M theory interesting mathematically and provide useful tools for addressing problems in string phenomenology (scanning the landscape) and gauge/gravity dualities

**In the context of the gauge/gravity duality:**

• New perspectives on familiar examples

• Methods to address more complicated models

• Can lead to the discovery of new gauge/gravity duals
General supersymmetric geometries

- Supersymmetric geometries of $d = 11, 10$ and $d < 10$ supergravities may be analysed systematically in the framework of $G$-structures/generalised geometry

**Input**: general metric ansatz plus Killing spinors

**Output**: set of equations for RR+NS fluxes and (multi-) forms
Supersymmetric geometries of $d = 11, 10$ and $d < 10$ supergravities may be analysed systematically in the framework of $\mathbb{G}$-structures/generalised geometry

**Input:** general metric ansatz plus Killing spinors

**Output:** set of equations for RR+NS fluxes and (multi-)forms

In Type IIB supergravity:

\[
\begin{align*}
    ds^2 &= e^{2\Delta} \left[ dx_{1,3}^2 + ds_6^2 \right] \\
    F_5 &= e^{4\Delta + \Phi} (1 + \ast_{10}) \text{vol}_4 \wedge f \\
    F_1 &= 0 \quad (\text{for simplicity})
\end{align*}
\]
Type IIB supersymmetric geometries

\[ e^{-2\Delta+\Phi/2}(d-H_3\wedge)e^{2\Delta-\Phi/2}\Psi_1 = d(\Delta + \frac{\Phi}{4}) \wedge \bar{\Psi}_1 + \frac{ie^{\Delta+5\Phi/4}}{8} [f - \ast_6 F_3] \]

\[ (d-H_3\wedge)e^{2\Delta-\Phi/2}\Psi_2 = 0 \]

[Graña, Minasian, Petrini, Tomasiello]

- \( \Psi_1, \Psi_2 \) are “pure spinors” in the sense of generalised geometry. Alternatively: multi-forms

- We restrict to the case when these take the form

\[ \Psi_1 = -e^{i\zeta} e^{\Delta+\Phi/4} \left( 1 - ie^{2\Delta+\Phi/2} J - \frac{1}{2} e^{4\Delta+\Phi} J \wedge J \right) \]

\[ \Psi_2 = -e^{4\Delta+\Phi} \Omega \]

- \( J, \Omega \) define a more familiar SU(3) structure. Non-constant phase \( \zeta \) allows interpolation between different classes
Interpolating SU(3) structures

“Geometry”

\[ \text{d} \left( e^{6 \Delta + \Phi/2} \Omega \right) = 0 \]
\[ \text{d} \left( e^{8 \Delta} J \wedge J \right) = 0 \]
\[ \text{d} \left( e^{2 \Delta - \Phi/2} \cos \zeta \right) = 0 \]
Interpolating SU(3) structures

“Geometry”

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  d \left( e^{6\Delta + \Phi/2} \Omega \right) &= 0 \\
  d \left( e^{8\Delta} J \wedge J \right) &= 0 \\
  d \left( e^{2\Delta - \Phi/2} \cos \zeta \right) &= 0 
\end{align*}
\]

“Fluxes”

\[
\begin{align*}
  *_6 F_3 &= -e^{-2\Delta - 3\Phi/2} \sec \zeta d \left( e^{4\Delta + \Phi} J \right) \\
  H_3 &= -\sin \zeta e^\Phi *_6 F_3 \\
  f &= -e^{-4\Delta - \Phi} d \left( e^{4\Delta} \sin \zeta \right)
\end{align*}
\]
Interpolating SU(3) structures

“Geometry”

\[ d \left( e^{6\Delta + \Phi/2} \Omega \right) = 0 \]
\[ d \left( e^{8\Delta} J \wedge J \right) = 0 \]
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“Fluxes”

\[ *_6 F_3 = -e^{-2\Delta - 3\Phi/2} \sec \zeta \, d \left( e^{4\Delta + \Phi} J \right) \]
\[ H_3 = -\sin \zeta e^{\Phi} *_6 F_3 \]
\[ f = -e^{-4\Delta - \Phi} d \left( e^{4\Delta} \sin \zeta \right) \]

• \( \sin \zeta \rightarrow 1 \): warped Calabi-Yau with ISD 3-form \( dJ = d\Omega = 0 \), \( G_3 = F_3 + iH_3 = i \star G_3 \), \( e^{\Phi} = g_s \) [Giddings, Kachru, Polchinski]
  E.g.: Klebanov-Strassler

• \( \cos \zeta \rightarrow 1 \): “superstrings with torsion” [Strominger]
  E.g.: Maldacena-Núñez
Non-Kähler geometries (Type I)

"Geometry" ($\zeta = 0$)

\[
\begin{align*}
\text{d} \left( e^{2\Phi} \Omega \right) &= 0 \\
\text{d} \left( e^{2\Phi} J \wedge J \right) &= 0 \\
\Delta &= \Phi / 4
\end{align*}
\]
### Non-Kähler geometries (Type I)

#### "Geometry" ($\zeta = 0$)

- $d \left( e^{2\Phi} \Omega \right) = 0$
- $d \left( e^{2\Phi} J \wedge J \right) = 0$
- $\Delta = \Phi / 4$

#### "Fluxes" ($\zeta = 0$)

- $\ast_{6}F_{3} = -e^{-2\Phi} d \left( e^{2\Phi} J \right)$
- $H_{3} = 0$
- $F_{5} = 0$

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- **S-dual version** involves only: metric, dilaton, NS 3-form $H_{3}$ → Type I/Heterotic solutions

- **$M_{6}$** is complex but non-Kähler. Killing spinors preserved by connection $\hat{\nabla} = \nabla_{\text{spin}} + H_{3}$ with **torsion**

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[Gauntlett,DM,Waldram]
Generating new solutions

[Minasian, Petrini, Zaffaroni], [Gaillard, DM, Núñez, Papadimitriou]

Simple solution generating method

In: solution to “non-Kähler” equations → out: general solution
Generating new solutions

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Simple solution generating method

\textbf{In:} solution to “non-Kähler” equations $\rightarrow$ \textbf{out:} general solution

\[
\Phi^{\text{new}} = \Phi^{\text{old}}, \quad \sin \zeta = \kappa_2 e^{\Phi^{\text{old}}}
\]
\[
e^{2\Delta} = \frac{\kappa_1}{\cos \zeta} e^{\Phi^{\text{old}}/2}, \quad F_3^{\text{new}} = \frac{1}{\kappa_1^2} F_3^{\text{old}}
\]

non trivial $F_5, H_3$ generated
Generating new solutions

[Minasian,Petrini,Zaffaroni], [Gaillard,DM,Núñez,Papadimitriou]

Simple solution generating method

\[ \Phi^{\text{new}} = \Phi^{\text{old}} \]
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non trivial \( F_5, H_3 \) generated

- Key point: Bianchi identity of simpler system \( \Rightarrow \) Bianchi of more general system \( \Rightarrow \) equations of motion (integrability results)

- Applies also with supersymmetric sources [Koerber,Tsimpis]

- Application to gauge/gravity duality: connection between wrapped fivebranes and Klebanov-Strassler theory
The conifold and the conifold transition

- The (Calabi-Yau) conifold singularity: \( z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \)

- Two desingularisations of the tip preserving the CY condition

  1. “Deformation”: \( \sum_i z_i^2 = \epsilon^2 \quad \text{T}^* \mathbb{S}^3 \cong \mathbb{S}^3 \times \mathbb{R}^3 \)

  2. “Resolution”: \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1 \cong \mathbb{S}^2 \times \mathbb{R}^4 \)

- [Vafa]: for large \( N \), \( N \) D5 branes wrapped on \( \mathbb{S}^2 \) in the resolved conifold \( \leftrightarrow \) deformed conifold with \( N \) units of RR \( F_3 \) through \( \mathbb{S}^3 \)

- Is it possible to see the geometric transition purely in the context of (Type IIB) supergravity?
The unwarped resolved deformed conifold

- $M$ fivebranes wrapped on the $S^2$ of the resolved conifold. Back-reaction of branes ($M$ large) modifies the geometry → work with “non-Kähler” equations
The unwarped resolved deformed conifold

- $M$ fivebranes wrapped on the $S^2$ of the resolved conifold. Back-reaction of branes ($M$ large) modifies the geometry → work with “non-Kähler” equations

- [Papadopoulos, Tseytlin] ansatz → back-reacted solution

$$ds^2_{\text{str}} = dx_{3+1}^2 + \frac{M}{4} ds^2_6$$

$$H_3 = \frac{M}{4} w_3$$

$$e^{2\Phi(t)} = e^{2\Phi_0} \frac{\sqrt{f(t)c(t)'}}{\sinh^2 t}$$

- $ds^2_6$ depends (simply) on a function $c(t)$. $w_3$ is a 3-form

- Solution explicit, up to 1st order ODEs:

$$f' = 4 \sinh^2 t c \quad c' = \frac{1}{f} [c^2 \sinh^2 t - (t \cosh t - \sinh t)^2]$$

- Parameters: $M, \Phi_0, 0 < U < \infty$. $U$ is defined at large $t$ and matched (numerically) to a parameter $\gamma^2 \geq 1$ near $t \sim 0$
Realising the geometric transition

- $t \to 0$: $r^2_{S^3} \sim M \gamma^2 \to$ radius of $S^3$
- $t \to \infty$: $r^2_{S^2} \sim M \log U \to$ radius of $S^2$

- Parameter $U$ interpolates between deformation and resolution

- $U \to 0$: $\approx$ deformed conifold with large $S^3 + \int H_3 = M$ flux
- $U \to \infty$: $\approx$ resolved conifold + $M$ NS5 branes (far from branes)
Field theory (decoupling) limit: large U

- Generalised GVW superpotential $\rightarrow$ define a “gauge coupling”:
  $$W = \int_{M_6} e^{-2\Phi} \Omega \wedge (H_3 + \text{id}J) \implies \beta_{8\pi^2} = 3M$$

- Decoupling limit (near brane): $U \rightarrow \infty \implies$ field theory
  $$\lambda'_{\text{Hooft}} = g_Y^2 M \sim \frac{1}{\log U} \ll 1$$

$\Rightarrow$ [Maldacena,Núñez] (CV-MN) solution: $SU(M) \mathcal{N} = 1$ SYM
The baryonic branch from fivebranes

- Using the generating technique, we can add D3 branes and B-field to the “unwarped resolved deformed conifold”

- Warp factor generated: \[ h = 1 + \cosh^2 \beta (e^{2(\Phi - \Phi_\infty)} - 1) \]

- Transformed solution has all fluxes (except \( F_1 \)) and depends on one new parameter (\( \beta \)): \( M, \Phi_\infty, U, \beta \)
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- Solution incorporates various limits. In a “decoupling limit” \( \beta \to \infty \) we recover the baryonic branch of the KS theory [Butti, Graña, Minasian, Petrini, Zaffaroni]
The baryonic branch from fivebranes

- Using the generating technique, we can add $D3$ branes and $B$-field to the “unwarped resolved deformed conifold”

- **Warp factor** generated: $h = 1 + \cosh^2 \beta (e^{2(\Phi - \Phi_\infty)} - 1)$

- Transformed solution has all fluxes (except $F_1$) and depends on one new parameter ($\beta$): $M, \Phi_\infty, U, \beta$

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New perspective on the baryonic branch of Klebanov-Strassler

Fivebranes on $S^2$ of resolved conifold + $B$-field: for large $U$ we expect fivebranes on a fuzzy $S^2$
Fuzzy two-sphere from the baryonic branch of KS

- Klebanov-Strassler theory: \( SU(M_k) \times SU(M(k+1)) \) quiver
Fuzzy two-sphere from the baryonic branch of KS

- Klebanov-Strassler theory: $\text{SU}(Mk) \times \text{SU}(M(k + 1))$ quiver

- Classical baryonic branch vacuum [Dymarsky, Klebanov, Seiberg]

\[
A_i = C \Phi_i \otimes 1_{M \times M}, \quad B_i = 0
\]

\[
\Phi_1 = \begin{pmatrix}
\sqrt{k} & 0 & 0 & \cdots & 0 \\
0 & \sqrt{k-1} & 0 & \cdots & 0 \\
0 & 0 & \sqrt{k-2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\]

\[
\Phi_2 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & \sqrt{2} & \cdots & 0 & 0 \\
0 & 0 & 0 & \sqrt{3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \sqrt{k}
\end{pmatrix}
\]

D - terms:

\[
\begin{align*}
A_1 A_1^\dagger &+ A_2 A_2^\dagger - B_1 B_1^\dagger - B_2 B_2^\dagger = (k + 1) |C|^2 1_k \\
A_1^\dagger A_1 &+ A_2^\dagger A_2 - B_1 B_1^\dagger - B_2 B_2^\dagger = k |C|^2 1_{k+1}
\end{align*}
\]
Fuzzy two-sphere from the baryonic branch of KS

- Klebanov-Strassler theory: \( SU(M_k) \times SU(M(k + 1)) \) quiver

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0 & 0 & \sqrt{k-2} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 \\
\end{pmatrix}
\]

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\Phi_2 = \begin{pmatrix}
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0 & 0 & 0 & \sqrt{3} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & \sqrt{k} \\
\end{pmatrix}
\]

D-terms:

\[
\begin{aligned}
A_1 A_1^\dagger + A_2 A_2^\dagger - B_1^\dagger B_1 - B_2^\dagger B_2 &= (k + 1) |C|^2 1_k \\
A_1^\dagger A_1 + A_2^\dagger A_2 - B_1 B_1^\dagger - B_2 B_2^\dagger &= k |C|^2 1_{k+1}
\end{aligned}
\]

- From the \( k \times (k + 1) \) matrices \( \Phi_i \) we construct matrices spanning two irreducible representations of \( SU(2) \)
Fuzzy two-sphere from the baryonic branch of KS

\[ L_1 = \frac{1}{2} (\Phi_1 \Phi_2^\dagger + \Phi_2 \Phi_1^\dagger) \]
\[ L_2 = \frac{i}{2} (\Phi_1 \Phi_2^\dagger - \Phi_2 \Phi_1^\dagger) \]
\[ L_3 = \frac{1}{2} (\Phi_1^\dagger \Phi_1 - \Phi_2^\dagger \Phi_2) \]

Define a spin \( j = \frac{k - 1}{2} \) irreducible representation of \( SU(2) \)

\[ R_1 = \frac{1}{2} (\Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1) \]
\[ R_2 = \frac{i}{2} (\Phi_2^\dagger \Phi_1 - \Phi_1^\dagger \Phi_2) \]
\[ R_3 = \frac{1}{2} (\Phi_1^\dagger \Phi_1 - \Phi_2^\dagger \Phi_2) \]

Define a spin \( j = \frac{k}{2} \) irreducible representation of \( SU(2) \)

- Looks like these define \( SU(2) \times SU(2) \cong SO(4) \), but in fact they define a fuzzy super two-sphere
The fuzzy sphere spectrum (weak coupling)

- Fluctuations: $A_i = \Phi_i + \delta A_i, \quad B_i = \delta B_i, \quad a^{L,R}_\mu = \delta a^{L,R}_\mu$

<table>
<thead>
<tr>
<th>fields</th>
<th>on $S^2$</th>
<th>SU(2) spin</th>
<th>$\mathcal{N'} = 1$ multiplet</th>
<th>eigenvalues</th>
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</thead>
<tbody>
<tr>
<td>$a^L_\mu, a^R_\mu$</td>
<td>scalar</td>
<td>$j = l$</td>
<td>1 vector</td>
<td>$\lambda_l, -, \lambda_l, +$</td>
</tr>
<tr>
<td>$\delta A_i$</td>
<td>vector</td>
<td>$j = l$</td>
<td>1 vector</td>
<td>$\lambda_l, -, \lambda_l, +$</td>
</tr>
<tr>
<td>$B_i$</td>
<td>spinor</td>
<td>$j = l + \frac{1}{2}$</td>
<td>2 chiral</td>
<td>$\lambda_l, B$</td>
</tr>
</tbody>
</table>

- Eigenvalues for $l \ll k$

$$\lambda_{l,-} \sim \frac{g_+^2|C|^2}{2k + 1} l(l + 1), \quad \lambda_{l,B} = |C|^4 h^2(l + 1)^2$$

- Agrees with spectrum of Maldacena-Núñez compactification of D5 branes wrapped on $S^2$ [Andrews, Dorey]

Parameters: $\theta_{\text{Fuzzy}} \propto \frac{1}{k}, \quad |C|^2 R^2_{\text{Fuzzy}} \propto \frac{k}{g^2_+}$
Comparison with gravity (strong coupling)

- Compare with the parameters computed in the gravity solution (in the intermediate “fivebrane” region)

\[ \int_{S^2} B \propto g_s M \log U \equiv k \quad \# \text{ cascade steps} \]

\[ \langle U \rangle \sim \sum_{i=1,2} \text{Tr}[A_i A_i^\dagger - B_i^\dagger B_i] \sim |C|^2 \propto MU \Lambda_0^2 \]

[Dymarsky, Klebanov, Seiberg]

- Large \( B \)-field \( \Rightarrow \) use open string metric: \( r_{\text{open}} \sim \frac{B}{r_{\text{closed}}} \)

[Seiberg, Witten]

Parameters:

\[ \theta_{\text{NC}} \sim \frac{1}{B} \quad \frac{m_{\text{KK}}^2}{|C|^2} \sim \frac{g_+^2}{k} \]
Adding flavours

[Gaillard,DM,Núñez,Papadimitriou]

- Branes wrapped on an infinitely extended surface $\rightarrow$ effective 4d coupling constant vanishes $\rightarrow$ “flavours” [Karch,Katz]

- Back-reacted solution with $N_f \sim N_c$ smeared D5 “flavour branes” constructed in [Casero,Nuñez,Paredes]

- $\text{SU}(3)$ structure transformation $\rightarrow$ “flavoured warped resolved deformed conifold”, includes: $N_c$ “colour D5”, $N_f$ “flavour D5”, plus bulk and source D3 branes

- Different from previous “flavoured” solutions, obtained with D7 branes. Possible because D5 probes (with D3 charge) are supersymmetric on the “resolved deformed conifold”
The fivebranes set up

<table>
<thead>
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<td>$S_{\text{base}}^2$</td>
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<td>$N_c$ D5</td>
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</table>

- Directions **456789** span a resolved conifold: $\mathbb{R}^4 \rightarrow S_{\text{base}}^2$.
- $S_{\text{fiber}}^2 \subset \mathbb{R}^4$ fiber

- To include back-reaction use non-Kähler equations: $F_3$, $\Phi$, $g_{\mu\nu}$

$$S = S_{\text{type I}} - \int \left( e^{6\Delta+\Phi/2} \text{vol}(4) \wedge J - C(6) \right) \wedge \Xi(4)$$

- $N_f$ D5 branes smeared on **4567** $\rightarrow$ smearing form $\Xi(4)$

- Susy (non-Kähler) equations unchanged. Bianchi identity $\rightarrow dF_3 = \Xi(4)$
• Ansatz unchanged. BPS equations slightly modified:

\[ f' = 4 \sinh^2 t \ c \quad c' = \frac{1}{f} [c^2 \sinh^2 t - (t \cosh t - \sinh t)^2] - \frac{N_f}{2N_c} \]

• Parameters of the solution: \( N_f, N_c, e^{\Phi_\infty} = g_s, U \)

• UV asymptotics unchanged. E.g. \( \Phi \rightarrow \text{constant} \)

• Singularity in the IR. We impose boundary conditions such that the smooth solution is recovered for \( N_f \rightarrow 0 \)
The flavoured fivebrane field theory

- In the decoupling limit $U \to \infty$ there is a field theory dual

- $\text{SU}(N_c) \text{SQCD} + N_f$ quarks $q_i, \tilde{q}_i$ [Casero, Nuñez, Paredes]

- Quartic superpotential $W \sim (q\tilde{q})^2$

- Beta function $\beta = 3 (N_c - N_f/2)$

- Smearing on $S^2_{45} \times S^2_{67} \to$ extra $\text{SU}(2) \times \text{SU}(2)$ symmetry

- $\text{SU}(N_c) \times [\text{SU}(N_f/2) \times \text{SU}(2) \times \text{SU}(2)]_{\text{flavour}} \times \text{U}(1)_R$ ($\text{U}(1)_R$ broken to $\mathbb{Z}_2$)
The flavoured, deformed, resolved, conifold

- Using the transformation we add D3 branes, B-field and generate a warp factor \( h = 1 + \cosh^2 \beta (e^{2(\Phi - \Phi_\infty)} - 1) \)

- In decoupling limit \( \beta \to \infty \) we obtain new solution generalising that of [Butti et al] by the addition of flavour D5 branes
The flavoured, deformed, resolved, conifold

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- A distribution of source D3 branes is generated (as required by $\kappa$-symmetry)

$$ h \sim \frac{(N_f/U) r^2 + (N_c - N_f/2)^2 \log r}{r^4} + O(1/r^4) $$

- Define two types of running number of D3 branes

$$ n_f \sim \frac{N_f}{U} r^2 \equiv \nu r^2 \quad n \sim (N_c - N_f/2)^2 \log r \quad \text{for} \quad r \to \infty $$
Klebanov-Strassler + source D3 branes

• Instructive to consider a special limit: $U \to 0$

• In the unflavoured [Butti et al] solution this gives the ordinary Klebanov-Strassler solution

• With $N_f \neq 0$ this limit is not possible. Reason: probe D5 branes are not susy on the deformed conifold

• Consider a new limit: $U \to 0, N_f \to 0$ at fixed $N_f/U = \nu$. The internal metric is the deformed conifold, warped by:

\[ h = 2\nu \int_0^\infty d\rho' \left( \sinh(4\rho') - 4\rho' \right)^{-1/3} + h_{KS} \]

\[ dF_5 - H_3 \wedge F_3 = \nu B \wedge \Xi(4) \neq 0 \quad \Rightarrow \quad \text{source D3 branes} \]
Comments on the field theory

- UV asymptotics $\Rightarrow$ would-be Goldstone boson mode for spontaneously broken $U(1)_B$ is not normalisable

- Indeed presence of $n_f$ extra D3 branes suggests changing the theory from (1) $SU(N_c(k + 1)) \times SU(kN_c)$ to
  
  $$(2) \quad SU(N_c(k + 1) + n_f) \times SU(kN_c + n_f)$$

- In (1) there is a baryonic branch, whereas in (2) there isn't $\rightarrow$ theory on the mesonic branch! [Dymarsky,Klebanov,Seiberg]

- Two different types of varying $F_5$ flux interpreted as:
  - $n \sim N_c^2 \log r \rightarrow$ cascade of Seiberg dualities
  - $n_f \sim \nu r^2 \rightarrow$ Higgsing [Aharony]

- Adding back $N_f$ complicates the picture in an interesting + puzzling way..
A $G_2$ story

- Flash out a similar construction in Type IIA supergravity

\[ ds^2 = h^{-1/2} dx_{1,2}^2 + h^{1/2} ds_7^2 \]

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A $G_2$ story

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“Geometry”

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<table>
<thead>
<tr>
<th>“Geometry”</th>
<th>“Fluxes”</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d(e^{-2\Phi} \ast_7 \phi) = 0$</td>
<td>$F_4 = \text{vol}_3 \wedge dh^{-1} + c_2 d(e^{-2\Phi} \phi)$</td>
</tr>
<tr>
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<td>$\ast_7 H_3 = c_1 e^{2\Phi} d(e^{-2\Phi} \phi)$</td>
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- $\zeta \rightarrow 0$: 7d counterpart of “non-Kähler” geometries [GMPW]

- $\zeta \rightarrow \pi/2$: warped $G_2$ holonomy manifold [Cvetic, Lu, Pope]
A $G_2$ story

- **Solution generating method**: start with $M$ fivebranes wrapped on the $S^3$ inside the $G_2$-manifold $X = S^3 \times \mathbb{R}^4$. After backreaction the geometry is "$G_2$ with torsion": there is an interpolating parameter $U$.

- $U \rightarrow 0$: $G_2$-manifold $X$ with large $S^3 + H_3$ flux

- $U \rightarrow \infty$: $G_2$-manifold $\tilde{X} + NS5$ branes on $\tilde{S}^3$

- Realises $G_2$ geometric transition in Type IIA supergravity

- Decoupling limit (near brane): $U \rightarrow \infty \Rightarrow$ field theory
  \[ \Rightarrow \text{T-dual [Maldacena,Nastase]: } SU(M)_{\frac{M}{2}} \mathcal{N} = 1 \text{ Chern-Simons} \]

- $\mathcal{N} = 1$, 3d field theory dual to warped $G_2$ manifold not known
  \[ \rightarrow \text{presumably it is a Chern-Simons theory} \]
Outlook

• **Solution generating transformation** for classes of supersymmetric geometries of Type IIA/IIB

• Solutions realising **geometric transitions** in Type IIB (torsional $\text{SU}(3)$) and Type IIA (torsional $\text{G}_2$)

• Perhaps there exist other classes with similar features, besides $\text{SU}(3)$ and $\text{G}_2$. Eleven dimensions?

• Relation between **baryonic branch** of KS and **fuzzy two-sphere** may be explored for more general quiver theories

• Constructed new **flavoured resolved deformed conifold** solution. Field theory interpretation seems different from baryonic branch (and any other solutions so far)

• The $\text{G}_2$ story: the geometry works as in the $\text{SU}(3)$ case. It would be nice to have a field theory picture