Full-sky gravitational lensing: new relativistic effects in convergence and second-order shear

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Weak lensing

Lensing describes the **deflection of light** by the gravitational potential → it is sensitive to the total distribution of matter in the Universe.

It is a powerful tool to map the large-scale structure.

The distortion created by weak lensing can be split in two quantities: the convergence and the shear.
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Observations

Shear

Correlations between the ellipticity of galaxies are measured.
First measurements in 2000. Bacon et al., Kaiser et al., Wittman et al., van Waerbeke et al.
Recently with CFHTLS. Fu et al. 2008.

Convergence

The intrinsic size of the galaxies is unknown.
The number of galaxies per unit of solid angle as a function of flux and redshift is known. This number is affected by the convergence. Schneider et al. 1992, Bartelmann 1995, Broadhurst 1995
Future surveys

Pan-STARRS, DES, LSST, JDEM, Euclid

Improvements

♦ A better statistic

♦ Accurate measurements of the source redshift

♦ Nearly all-sky surveys (Euclid: 20‘000 square degrees)

This will open the way to new type of studies.
New constraints

- Redshift information allows cosmic shear tomography → measure the evolution of large-scale structure.

- Redshift information allows precise measurement of the convergence. Zhang and Pen 2006, 1% accuracy with the SKA → complementary results since the convergence is not affected by the same uncertainties + new relativistic effects like the one due to peculiar velocities of galaxies. Bonvin 2008

- Nearly all-sky surveys allow to measure correlations at large scales → uncover new relativistic effects at second order in the shear. Bernardeau, Bonvin and Vernizzi 2010
Outline

♦ Relativistic calculation of the convergence and of the shear.

♦ Calculation at first order
  • Standard formula for the shear.
  • New effect in the convergence: peculiar velocities are important for redshift smaller than one.
  • The consistency relation is modified and can be used to measure peculiar velocities.

♦ Calculation at second order
  • Standard terms dominating at small scales.
  • New contributions relevant for all-sky surveys.
  • Compute the bispectrum.
Sachs equation

\[
\frac{D^2 \xi^\alpha (\lambda)}{D \lambda^2} = R^\alpha_{\beta \mu \nu} k^\beta k^\mu \xi^\nu
\]

Evolution equation describing the distortion of the beam.

We project the equation on a basis \( \{ v_\alpha^O, k^\alpha, n_1^\alpha, n_2^\alpha \} \)

\[
\frac{d^2 \xi^a}{d\lambda^2} = R_{ab} \xi^b \quad \text{with} \quad R_{ab} = R_{\alpha \beta \mu \nu} k_\beta^\mu k_\mu^\nu n_1^\alpha n_2^\nu
\]

Solution: \( \xi^a_O = D_{ab} \theta^b_O \quad \theta^b_O = \xi^b_O' \)

\( D_{ab} \) magnification matrix
Magnification matrix

\[ \frac{d^2}{d\lambda^2} \mathcal{D}_{ab} = \mathcal{R}_{ac} \mathcal{D}_{cb} \quad \text{with} \quad \mathcal{R}_{ab} = R_{\alpha\beta\mu\nu} k^\beta k^\mu n^\alpha_a n^\nu_b \]

Convergence
\[ \kappa = - \frac{1}{2\lambda_S} (\mathcal{D}_{11} + \mathcal{D}_{22}) \]

Shear
\[ \gamma = - \frac{1}{2\lambda_S} \left[ \mathcal{D}_{11} - \mathcal{D}_{22} + i(\mathcal{D}_{12} + \mathcal{D}_{21}) \right] \]

Rotation
\[ \omega = - \frac{1}{2\lambda_S} (\mathcal{D}_{12} - \mathcal{D}_{21}) \]

Observable quantities
\[ g = \frac{\gamma}{1 - \kappa} \]
\[ \delta_g = \frac{1}{\left[ (1 - \kappa)^2 + \gamma \gamma^* \right]^{\alpha - 1}} - 1 = 2(\alpha - 1)\kappa \]
First order

\[ ds^2 = -(1 + 2\phi)d\eta^2 + (1 - 2\psi)dx^2 \]

The effect of the expansion \( D_{ab} \to a_S D_{ab} \)

\[ \gamma(\chi_S) = \int_0^{\chi_S} d\chi \frac{\chi_S - \chi}{\chi \chi_S} \phi^2 \Psi \]

\[ \kappa(\chi_S) = \psi(\chi_S) + \int_0^{\chi_S} d\chi \left( -\frac{2}{\chi_S} \Psi + \frac{\chi_S - \chi}{\chi \chi_S} \phi \bar{\phi} \Psi \right) \]

Consistency relation \( C_{\ell}^\kappa = \frac{\ell(\ell + 1)}{(\ell + 2)(\ell - 1)} C_{\ell}^\gamma \)  

Hu 2000
First order

\[ ds^2 = -(1 + 2\phi) d\eta^2 + (1 - 2\psi) dx^2 \]

\[ \Psi = \frac{1}{2}(\psi + \phi) \quad \chi = \eta_0 - \eta \]

\[ \phi = -\sin^s \theta \left[ \partial_\theta + \frac{i}{\tan \theta} \partial_\varphi \right] (\sin^{-s} \theta) \]

\[ \gamma(\chi_S) = \int_0^{\chi_S} d\chi \frac{\chi_S - \chi}{\chi \chi_S} \phi^2 \Psi \]

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\[ C^\kappa_\ell = \frac{\ell(\ell + 1)}{(\ell + 2)(\ell - 1)} C^\gamma_\ell \quad \text{Hu 2000} \]
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\[ \gamma(\chi S) = \int_0^{\chi S} d\chi \left( \frac{\chi S - \chi}{\chi \chi S} \right) \ddot{\phi} \Psi \]

\[ \kappa(\chi S) = \psi(\chi S) + \int_0^{\chi S} d\chi \left( -\frac{2}{\chi S} \Psi + \frac{\chi S - \chi}{\chi \chi S} \ddot{\phi} \dddot{\phi} \Psi \right) \]

Consistency relation

\[ C^\kappa_\ell = \frac{\ell(\ell + 1)}{(\ell + 2)(\ell - 1)} C^\gamma_\ell \quad \text{Hu 2000} \]

\( \dot{\phi} \Psi \quad \text{Spin one} \)

\( \ddot{\phi} \Psi \quad \text{Spin two} \)
Redshift perturbations

The **conformal time** $\eta_S$ is not an observable quantity. We need to express $D_{ab}$ as a function of the **redshift** $z_S$.

In a perturbed Universe, the redshift is perturbed.

- At first order, only the **convergence** is affected.
- At second order, the **shear** is also affected.
Effect on the convergence

\[ \delta z_S = (1 + z_S) \left( \phi_O - \phi_S + \mathbf{n} \cdot (\mathbf{v}_O - \mathbf{v}_S) + 2 \int_0^{\chi_S} d\chi \dot{\Psi} \right) \]

\[ \kappa_\Psi = \int_0^{\chi_S} d\chi \frac{\chi_S - \chi}{\chi \chi_S} \frac{\phi}{\dot{\phi}} \dot{\Psi} \]

\[ \kappa_v = \left( 1 - \frac{1}{H_S \chi_S} \right) \mathbf{n} \cdot (\mathbf{v}_O - \mathbf{v}_S) \]

The two contributions affect the overdensity of galaxies.

\[ \delta_g = 2(\alpha - 1)(\kappa_\Psi + \kappa_v) \]

The parameter \( \alpha \) comes from the modelling of the unlensed galaxy number density.
Angular power spectra

We expand $\delta_g$ in spherical harmonics, and we determine the **angular power spectrum**.

$$\langle \delta_g(z_S, n) \delta_g(z_S, n') \rangle = \sum_\ell \frac{2\ell + 1}{4\pi} C_\ell(z_S) P_\ell(n \cdot n')$$

The angular power spectrum contains two contributions.

$$C_\ell = C_\ell^\Psi + C_\ell^v$$

We choose a gaussian primordial power spectrum for $\Psi$ and we use Einstein’s equations to relate the velocity to the potential.
Results

Bonvin 2008

Standard contribution vs. Velocity contribution

\( l(1+1)C_{\ell}/2\pi \)

- \( z = 0.2 \)
- \( z = 0.5 \)
- \( z = 1 \)
- \( z = 0.7 \)
- \( z = 1.2 \)
- \( z = 1.5 \)
Consistency relation

\[ C_\ell^{\kappa \Psi} = \frac{\ell(\ell + 1)}{(\ell + 2)(\ell - 1)} C_\ell^\gamma \]  

Hu 2000

Peculiar velocities modify the convergence at first order, but they do not affect the shear; they modify the \textbf{consistency relation} between the observed convergence and the observed shear:

\[ C_\ell^{\kappa \Psi} = C_\ell^{\kappa \text{obs}} - \frac{\ell(\ell + 1)}{(\ell + 2)(\ell - 1)} C_\ell^\gamma \text{obs} \]

By measuring the convergence and the shear, we can measure peculiar velocities of galaxies.
Second order

\[ \frac{d^2}{d\lambda^2} D_{ab} = R_{ac} D_{cb} \]

\[ R_{ab} = R_{\alpha\beta\mu\nu} k^\beta k^\mu n^\alpha n^\nu \]

♦ We integrate on **perturbed geodesics** beyond Born approximation.

♦ We take into account **vector** and **tensor** modes

\[ ds^2 = -e^{2\phi} d\eta^2 + 2\omega_i d\eta dx^i + (e^{-2\psi} \delta_{ij} + h_{ij}) dx^i dx^j \]

♦ We compute the **reduced shear**: \[ g = \frac{\gamma}{1 - \kappa} \]

♦ We take into account the perturbations of the **redshift**.
Results: standard corrections

- Corrections to Born approximation

\[ \int_0^{\chi_s} d\chi \frac{1}{\chi \chi_s} \phi^3 \Psi \int_0^{\chi} d\chi' \frac{\chi - \chi'}{\chi'} \phi \Psi \]

\[ \ell^3 \times \ell = \ell^4 \]

- Lens-lens couplings

\[ \int_0^{\chi_s} d\chi \frac{1}{\chi \chi_s} \phi^2 \Psi \int_0^{\chi} d\chi' \frac{\chi - \chi'}{\chi'} \phi \phi \Psi \]

Shear  Convergence

References:
- Bernardeau et al. 1997
- Cooray and Hu 2002
- Dodelson et al. 2005
- Shapiro and Cooray 2006
New terms

- Intrinsic contribution

\[ \int_0^{\chi_S} d\chi \frac{\chi_S - \chi}{\chi \chi_S} \frac{\phi^2}{\partial} \Psi^2(\chi) \]

- Time delay-lens coupling

\[ \int_0^{\chi_S} d\chi \frac{1}{\chi_S} \Psi(\chi) \int_0^\chi d\chi' \frac{1}{\chi'} \frac{\phi^2}{\partial} \Psi(\chi') \]

- Deflection-displacement couplings

\[ \int_0^{\chi_S} d\chi \frac{\chi_S - \chi}{\chi \chi_S} \phi \Psi(\chi) \int_0^\chi d\chi' \frac{\chi - \chi'}{\chi'} \phi \Psi(\chi') \]
New terms

- Intrinsic contribution
  \[ \int_0^{\chi_s} d\chi \frac{\chi_s - \chi}{\chi \chi_s} \dot{\phi}^2 \Psi^2(\chi) \]

- Time delay-lens coupling
  \[ \int_0^{\chi_s} d\chi \frac{\chi_s - \chi}{\chi \chi_s} \dot{\Psi}(\chi) \int_0^\chi d\chi' \dot{\phi}^2 \Psi(\chi') \]

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  \[ \int_0^{\chi_s} d\chi \frac{\chi_s - \chi}{\chi \chi_s} \phi \Psi(\chi) \int_0^\chi d\chi' \frac{\chi - \chi'}{\chi'} \phi \Psi(\chi') \]
New terms

- **Intrinsic contribution**
  \[
  \int_0^{\chi_s} d\chi \frac{\chi_s - \chi}{\chi \chi_s} \phi^4 \Psi^2(\chi)
  \]

- **Time delay-lens coupling**
  \[
  \int_0^{\chi_s} d\chi \frac{1}{\chi \chi_s} \phi^2 \Psi(\chi) \int_0^\chi d\chi' \Psi(\chi')
  \]

- **Deflection-displacement couplings**
  \[
  \int_0^{\chi_s} d\chi \frac{\chi_s - \chi}{\chi \chi_s} \phi \Psi(\chi) \int_0^\chi d\chi' \frac{\chi - \chi'}{\chi'} \phi \Psi(\chi')
  \]
New terms

♦ Reduced shear

\[
\int_0^{\chi_S} d\chi \frac{\chi_S - \chi}{\chi \chi_S} \phi \bar{\phi} \Psi \int_0^{\chi_S} d\chi' \frac{\chi_S - \chi'}{\chi' \chi_S} \phi^2 \Psi - 2 \int_0^{\chi_S} d\chi \Psi \int_0^{\chi_S} d\chi' \frac{\chi_S - \chi'}{\chi' \chi_S^2} \phi^2 \Psi
\]

♦ Redshift perturbations

\[
\frac{1 + z_S}{\chi_S^2 H_S} \left( \phi(\chi_S) + \mathbf{n} \cdot \mathbf{v}_S - 2 \int_0^{\chi_S} d\chi' \dot{\Psi} \right) \int_0^{\chi_S} d\chi' \phi^2 \Psi
\]

♦ Vector and tensor

\[
\frac{1}{2} h(\chi_S) + \int_0^{\chi_S} d\chi \left[ \frac{\chi_S - \chi}{\chi \chi_S} \phi^2 (\omega_r + \frac{1}{2} h_{rr}) + \frac{1}{\chi} \phi (1 \omega + 1 h_r) \right]
\]
Weyl potential

Apart from the redshift corrections, all the terms depend only on the Weyl potential \( \Psi = \frac{1}{2}(\phi + \psi) \)

The metric can be rewritten as

\[
ds^2 = e^{-2\psi} \left[ -e^{2\Psi} \, d\eta^2 + 2\omega_i \eta d\xi^i + (\delta_{ij} + h_{ij}) \, d\xi^i \, d\xi^j \right]
\]

The conformal factor rescales the magnification matrix

\[
D_{ab} \rightarrow e^{-\psi} \cdot D_{ab}
\]

This has no impact on the reduced shear, since it is the ratio between the traceless part and the trace of the matrix.
Bispectrum

Bernardeau, Bonvin, Van de Rijt and Vernizzi, in preparation.

\[ g(z_S, \mathbf{n}) = \sum_{\ell m} 2a_{\ell m}(z_S) \cdot 2Y_{\ell m}(\mathbf{n}) \]

**E-modes**

\[ E(z_S, \mathbf{n}) = -\frac{1}{2} \sum_{\ell m} (2a_{\ell m} + -2a_{\ell m})Y_{\ell m}(\mathbf{n}) \]

**B-modes**

\[ B(z_S, \mathbf{n}) = \frac{i}{2} \sum_{\ell m} (2a_{\ell m} - -2a_{\ell m})Y_{\ell m}(\mathbf{n}) \]

At first order, the shear contains only E-modes. At second order it contains both.

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At second order it contains both.

\[ \langle E(z_S, \mathbf{n}_1)E(z_S, \mathbf{n}_2)E(z_S, \mathbf{n}_3) \rangle = \]

\[ \sum_{\ell_1\ell_2\ell_3 m_1 m_2 m_3} \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{array} \right) B_{\ell_1\ell_2\ell_3}(z_S)Y_{\ell_1 m_1}(\mathbf{n}_1)Y_{\ell_2 m_2}(\mathbf{n}_2)Y_{\ell_3 m_3}(\mathbf{n}_3) \]
Bispectrum

We computed \( B_{\ell_1 \ell_2 \ell_3}(z_S) \) associated with all the scalar non-linear corrections.

The overall *amplitude* and *time-dependence* is given by the particular form of the integrals.

The \( \ell \)-dependence is given by the transverse operators

\[
\Phi^2 \rightarrow \sqrt{\frac{(\ell + 2)!}{(\ell - 2)!}} \quad \quad \bar{\Phi} \Phi^2 \rightarrow \sqrt{\frac{(\ell + 1)!}{(\ell - 1)!}} (\ell - 1)(\ell + 2)
\]

We plot the bispectrum for galaxies at redshift \( z_S = 1 \)

We consider a flat gaussian primordial potential.
Squeezed configuration

\[ \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 2)(2\ell_3 + 1)}{4\pi}} \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{array} \right) \]

\[ \frac{b_{4\ell\ell}}{C_\ell C_\ell + 2C_\ell C_4} \]

Standard corrections:
- Born corrections
- Lens-lens couplings
- Reduced shear

\[ \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 2)(2\ell_3 + 1)}{4\pi}} \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{array} \right) \]

\[ \frac{b_{4\ell\ell}}{C_\ell C_\ell + 2C_\ell C_4} \]
Squeezed configuration

\[ \frac{b_{4\ell \ell+4}}{C_\ell C_{\ell+4} + C_\ell C_4 + C_{\ell+4} C_4} \]

Standard corrections:
• Born corrections
• Lens-lens couplings
• Reduced shear
Equilateral configuration

Standard corrections:
• Born corrections
• Lens-lens couplings
• Reduced shear

\[
\frac{b_{\ell\ell\ell\ell}}{3 \cdot C_\ell^2}
\]

\ell

red total
blue standard
green new
Primordial non-gaussianities: local

\[ \Psi = \Psi_L + f_{NL} \left( \Psi_L^2 - \langle \Psi_L^2 \rangle \right) \rightarrow \gamma = \int_0^{\chi_S} d\chi \frac{\chi_S - \chi}{\chi \chi_S} \theta^2 \Psi \]

Komatsu et al. 2009

\(-9 < f_{NL} < 111\)

\[ \frac{b_{4\ell\ell}}{C_\ell C_\ell + 2C_\ell C_4} \]

- red non-linearities
- blue local non-gaussianity \( f_{NL} = -10 \)
Primordial non-gaussianities: equilateral

Komatsu et al. 2009

\[-151 < f_{NL} < 253\]

\[
\frac{b_{\ell\ell\ell}}{3 \cdot C_{\ell}^2}
\]

### red non-linearities
### blue equilateral non-gaussianity $f_{NL} = 50$
Conclusion

- We presented a relativistic calculation of the convergence and of the shear, up to second order.

- We found new relativistic effects that may be relevant for future all-sky lensing surveys.

- Those terms affect the convergence at first order: in particular the peculiar velocities generate a new observable contribution.

- The shear is only affected at second order. We are currently computing the shear bispectrum associated with the non-linear corrections. In the squeezed limit, the relativistic effects seem important.