The Universe at Large Scales and the Renormalization Group

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• Motivations: precision cosmology, acoustic oscillations, and all that

• Eulerian Perturbation Theory: traditional and compact forms

• RG approach: formulation and first results. The emergence of an intrinsic UV cutoff
Tests of the **Perturbed FRW Universe**

Matter density: \( \rho(x, \tau) \equiv \bar{\rho}(\tau)[1 + \delta(x, \tau)] \)

Power spectrum: \( \langle \delta(k) \delta(k') \rangle = P(k)\delta^{(3)}(k - k') \)

Clumpiness \( \sim \) \( 4\pi k^3 P(k) \sim a^2 k^{3+n} \) (large scales, CDM)

perturbations get larger for smaller scales and larger times
A standard ruler: Baryonic Acoustic Oscillations

expect an excess probability of finding galaxies $\sim 100$ Mpc away from a given one

Found! 3.5 sigma evidence
Eisenstein et al. ‘05
Padmanabhan et al.

The same scale `seen’ in the CMB acoustic peaks, but at a much later epoch
A standard ruler: Baryonic Acoustic Oscillations

The same acoustic oscillation scale is imprinted on the CMB anisotropies.

Redshift surveys of galaxies (e.g. Sloan) measure this scale both along and across the line of sight.

Reconstruct the expansion history of the Universe from $z \approx 1000$ to today!
Future surveys

Goal: predict the LSS power spectrum to % accuracy

Ex.: BAO from WFMO
(2M galaxies at 0.5<z<1.)

peaks are a small effect.
require large surveys to detect.

Ex.: BAO from WFMO
(2M galaxies at 0.5<z<1.)

Goal: predict the LSS power spectrum to % accuracy
Non-linearities becomes more and more relevant in the DE-sensitive range $0 < z < 1$. 

Scoccimarro, '04

Jeong Komatsu, '06

1-loop PT

Present Status: Pert. Theory

$z = 0$

$z = 1$
Present Status: N-body simulations+fitting functions

\[ \Delta^2_{dm}(k) \]

\[ \text{Ratio} \]

\( k \text{ (h/Mpc)} \)

\( 0 \) to \( 0.5 \)

\( \Delta^2_{dm} (z=0) \)

PD96

Halo

Linear

Huff et al,'06

\(~10\%\) discrepancies between fitting functions and simulations

redshift-space distortions quite hard
Goals

• Improve Pert. Theory towards lower z and higher k

• Study the effect of non-linearities on baryonic acoustic oscillations
Dark Matter Hydrodynamics

The DM particle distribution function, \( f(x, p, \tau) \), obeys the Vlasov equation:

\[
\frac{\partial f}{\partial \tau} + \frac{p}{am} \cdot \nabla f - am \nabla \phi \cdot \nabla_p f = 0
\]

where \( p = am \frac{dx}{d\tau} \) and \( \nabla^2 \phi = \frac{3}{2} \Omega_M H^2 \delta \).

Taking momentum moments, i.e.,

\[
\int d^3p \ f(x, p, \tau) \equiv \rho(x, \tau) \equiv \bar{\rho}(\tau)[1 + \delta(x, \tau)]
\]

\[
\int d^3p \ \frac{p_i}{am} f(x, p, \tau) \equiv \rho(x, \tau)v_i(x, \tau)
\]

\[
\int d^3p \ \frac{p_i p_j}{a^2 m^2} f(x, p, \tau) \equiv \rho(x, \tau)v_i(x, \tau)v_j(x, \tau) + \sigma_{ij}(x, \tau)
\]

and neglecting \( \sigma_{ij} \) and higher moments (single stream approximation), one gets...
Equations of motion for single-stream cosmology

\[ \frac{\partial \delta}{\partial \tau} + \nabla \cdot [(1 + \delta)v] = 0, \quad \frac{\partial v}{\partial \tau} + \mathcal{H}v + (v \cdot \nabla)v = -\nabla \phi \]

In Fourier space, defining \( \theta(x, \tau) \equiv \nabla \cdot v(x, \tau) \),

\[ \frac{\partial \delta(k, \tau)}{\partial \tau} + \theta(k, \tau) + \int d^3 k_1 d^3 k_2 \delta_D(k - k_1 - k_2) \alpha(k_1, k_2) \theta(k_1, \tau) \delta(k_2, \tau) = 0 \]

\[ \frac{\partial \theta(k, \tau)}{\partial \tau} + \mathcal{H} \theta(k, \tau) + \frac{3}{2} \Omega_M \mathcal{H}^2 \delta(k, \tau) + \int d^3 k_1 d^3 k_2 \delta_D(k - k_1 - k_2) \beta(k_1, k_2) \theta(k_1, \tau) \theta(k_2, \tau) = 0 \]

mode-mode coupling controlled by:

\[ \alpha(k_1, k_2) \equiv \frac{(k_1 + k_2) \cdot k_1}{k_1^2} \]

\[ \beta(k_1, k_2) \equiv \frac{|k_1 + k_2|^2 (k_1 \cdot k_2)}{2k_1^2 k_2^2} \]
linear approximation: $\alpha(k_1, k_2) = \beta(k_1, k_2) = 0$

case one: no mode-mode coupling

\[
\frac{\partial \delta(k, \tau)}{\partial \tau} + \theta(k, \tau) = 0
\]

\[
\frac{\partial \theta(k, \tau)}{\partial \tau} + \mathcal{H} \theta(k, \tau) + \frac{3}{2} \Omega_M \mathcal{H}^2 \delta(k, \tau) = 0
\]

\[\Omega_M = 1 \rightarrow \mathcal{H} \sim a^{-1/2}\]

\[\delta(k, \tau) = \delta(k, \tau_i) \left(\frac{a(\tau)}{a(\tau_i)}\right)^m\]

\[\frac{\theta(k, \tau)}{\mathcal{H}} = m \delta(k, \tau)\]

\[m = \begin{cases} 
1 & \text{growing mode} \\
-\frac{3}{2} & \text{decaying mode}
\end{cases}\]
Assume EdS, $\Omega_M = 1$, then solutions have the form

$$\delta(k, \tau) = \sum_{n=1}^{\infty} a^n(\tau) \delta_n(k)$$

$$\theta(k, \tau) = -\mathcal{H}(\tau) \sum_{n=1}^{\infty} a^n(\tau) \theta_n(k)$$

with

$$\delta_n(k) = \int d^3q_1 \ldots d^3q_n \delta_D(k - q_1 \ldots n) F_n(q_1, \ldots, q_n) \delta_0(q_1) \ldots \delta_0(q_n)$$

$$\theta_n(k) = \int d^3q_1 \ldots d^3q_n \delta_D(k - q_1 \ldots n) G_n(q_1, \ldots, q_n) \delta_0(q_1) \ldots \delta_0(q_n)$$

The Kernels $F_n$ and $G_n$ satisfy recursion relations, with $F_1 = G_1 = 1$, and $\delta_1 = \theta_1 = \delta_0$:

$$F_n(q_1, \ldots, q_n) = \sum_{m=1}^{n-1} \frac{G_m(q_1, \ldots, q_m)}{(2n + 3)(n-1)}$$

$$\times [(2n+1)\alpha(k_1, k_2) F_{n-m}(q_{m+1}, \ldots, q_n) + 2\beta(k_1, k_2) G_{n-m}(q_{m+1}, \ldots, q_n)]$$

$$G_n(q_1, \ldots, q_n) = \ldots$$

where $k_1 = q_1 + \ldots + q_m$, $k_2 = q_{m+1} + \ldots + q_n$
An infinite number of basic vertices! very redundant!!
The hydrodynamical equations for density and velocity perturbations,

\[
\frac{\partial \delta}{\partial \tau} + \nabla \cdot [(1 + \delta)v] = 0, \quad \frac{\partial v}{\partial \tau} + Hv + (v \cdot \nabla)v = -\nabla \phi,
\]

can be written in a compact form (we assume an EdS model):

\[
(\delta_{ab} \partial_\eta + \Omega_{ab}) \varphi_b(\eta, k) = e^{\eta} \gamma_{abc}(k, -k_1, -k_2) \varphi_b(\eta, k_1) \varphi_c(\eta, k_2) \quad (1)
\]

where

\[
\left( \begin{array}{c} \varphi_1(\eta, k) \\ \varphi_2(\eta, k) \end{array} \right) = e^{-\eta} \left( \begin{array}{c} \delta(\eta, k) \\ -\theta(\eta, k)/H \end{array} \right) \quad \eta = \log \frac{a}{a_{in}} \quad \Omega = \left( \begin{array}{cc} 1 & -1 \\ -3/2 & 3/2 \end{array} \right)
\]

and the only non-zero components of the vertex are

\[
\gamma_{121}(k_1, k_2, k_3) = \gamma_{122}(k_1, k_3, k_2) = \delta_D(k_1 + k_2 + k_3) \frac{(k_2 + k_3) \cdot k_2}{2k_2^2}
\]

\[
\gamma_{222}(k_1, k_2, k_3) = \delta_D(k_1 + k_2 + k_3) \frac{|k_2 + k_3|^2 k_2 \cdot k_3}{2k_2^2 k_3^2}
\]
An action principle

Eq. (1) can be derived by varying the action

\[ S = \int d\eta_1 d\eta_2 \chi_a g_{ab}^{-1} \varphi_b - \int d\eta \, e^{\eta} \gamma_{abc} \chi_a \varphi_b \varphi_c \]

where the auxiliary field \( \chi_a(\eta, \mathbf{k}) \) has been introduced and \( g_{ab}(\eta_1, \eta_2) \) is the retarded propagator:

\[
(\delta_{ab} \partial_\eta + \Omega_{ab}) \, g_{bc}(\eta, \eta') = \delta_{ac} \delta_D(\eta - \eta')
\]

so that \( \varphi^0_a(\eta, \mathbf{k}) = g_{ab}(\eta, \eta') \varphi^0_b(\eta', \mathbf{k}) \) is the solution of the linear equation

Explicitly, one finds:

\[
g(\eta_1, \eta_2) = \begin{cases} 
B + A \, e^{-5/2(\eta_1 - \eta_2)} & \eta_1 > \eta_2 \\
0 & \eta_1 < \eta_2 
\end{cases}
\]

Growing mode

Decaying mode

Initial conditions:

\[
\varphi^0_b(\eta', \mathbf{k}) \propto u_b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ -3/2 \end{pmatrix}
\]

\[
B = \frac{1}{5} \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \\
A = \frac{1}{5} \begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix}
\]
A generating functional

The probability of the configuration \( \varphi_a(\eta_f) \), given the initial condition \( \varphi_a(\eta_i) \), is

\[
P[\varphi_a(\eta_f); \varphi_a(\eta_i)] = \delta[\varphi_a(\eta_f) - \overline{\varphi}_a[\eta_f; \varphi_a(\eta_i)]]
\]

Fixed extrema

\[
\sim \int \mathcal{D}'' \varphi_a \mathcal{D} \chi_b \exp \left\{ i \int_{\eta_i}^{\eta_f} d\eta \chi_a \left[ (\delta_{ab} \partial_\eta + \Omega_{ab}) \varphi_b - e^\eta \gamma_{abc} \varphi_b \varphi_c \right] \right\}
\]

only tree-level (saddle point)

The generating functional at fixed initial conditions is

\[
Z[J_a, \Lambda_b; \varphi_c(\eta_i)] = \int \mathcal{D} \varphi_a(\eta_f) \exp \left\{ i \int_{\eta_i}^{\eta_f} d\eta (J_a \varphi_a + \Lambda_b \chi_b) \right\} P[\varphi_a(\eta_f); \varphi_a(\eta_i)]
\]
We are interested in **statistical** correlations, not in single solutions:

\[
Z[J_a, \Lambda_b; K' s] = \int D\varphi_c(\eta_i) W[\varphi_c(\eta_i); K' s] Z[J_a, \Lambda_b; \varphi_c(\eta_i)]
\]

where all the initial correlations are contained in

\[
W[\varphi_c(\eta_i); K' s] = \exp \left\{ -\varphi_a(\eta_i; k) K_a(k) - \frac{1}{2} \varphi_a(\eta_i; k_a) K_{ab}(k_a, k_b) \varphi_b(\eta_i; k_b) + \cdots \right\}
\]

In the case of Gaussian initial conditions: \((K(k))^{-1}_{ab} = P^0_{ab}(k) \equiv u_a u_b P^0(k)\)

Putting all together...

\[
Z[J, \Lambda] = \int D\varphi D\chi \exp \left\{ \int d\eta_1 d\eta_2 \left[ -\frac{1}{2} \chi g^{-1} P^L g^T \chi + i \chi g^{-1} \varphi \right] - i \int d\eta \left[ e^{\eta} g^T \varphi - J \varphi - \Lambda \chi \right] \right\}
\]

where the initial conditions are encoded in the linear power spectrum: \(P^L_{ab}(\eta, \eta'; k) \equiv (g(\eta)P^0(k)g^T(\eta'))_{ab}\)

Derivatives of \(Z\) w.r.t. the sources \(J\) and \(\Lambda\) give all the N-point correlation functions (power spectrum, bispectrum, ...) and the full propagator (k-dependent growth factor)
All known results in cosmological perturbation theory are expressible in terms of diagrams in which only a trilinear fundamental interaction appears.
Beyond perturbation theory: the renormalization group

Inspired by applications of Wilsonian RG to field theory: the RG parameter is momentum

Modify the primordial power spectrum as: \( P_\lambda^0(k) = P^0(k) \Theta(\lambda - k) \) (step function)

then, plug it into the generating functional: \( Z[J, \Lambda] \longrightarrow Z_{\lambda}[J, \Lambda] \)

\[
Z_{\lambda}[J, \Lambda] = \int D\varphi D\chi \exp \left\{ \int d\eta_1 d\eta_2 \left[ -\frac{1}{2} \chi g^{-1} P_L^\lambda g^{-1} T + i \chi g^{-1} \varphi \right] - i \int d\eta [e^{\gamma} \chi \varphi - J \varphi - \Lambda \chi] \right\}
\]

this object generates all the N-point functions for the Universe in which there were no primordial perturbations with \( k > \lambda \)

The evolution from \( \lambda = 0 \) to \( \lambda = \infty \) can be described non-perturbatively by RG equations:

\[
\frac{\partial}{\partial \lambda} Z_{\lambda} = \int d\eta d\eta' \left[ \frac{1}{2} \frac{\partial}{\partial \lambda} \left( g^{-1} P_L^\lambda g^{-1} T \right)_{ab} \frac{\delta^2 Z_{\lambda}}{\delta \Lambda_b \delta \Lambda_a} \right]
\]
The propagator

\[ \delta^{(3)}(k + k') G_{\lambda,ab}(k; \eta_a, \eta_b) = - \frac{\delta^2 W_\lambda[J, \Lambda]}{\delta J_a(k, \eta_a) \delta \Lambda_b(k', \eta_b)} \]

\[ W_\lambda[J, \Lambda] = -i \log Z_\lambda[J, \Lambda] \]

\[ \frac{\partial}{\partial \lambda} \frac{\delta^2 W_\lambda}{\delta J_a \delta \Lambda_b} = \frac{1}{2} \int d\eta_c d\eta_d d^3q \delta(\lambda - q) \left( g^{-1} P^L g^{-1 \top} \right)_{cd} \frac{\delta^4 W_\lambda}{\delta J_a \delta \Lambda_b \delta \Lambda_c \delta \Lambda_d} \]

in pictures...

infinite tower of RGE's
**Approximation:** Full propagators, tree level vertices

\[
\frac{\partial}{\partial \lambda} G_{\lambda, ab}(k; \eta_a, \eta_b) =
\]

\[
4 \int d\eta_c d\eta_d d^3q \, \delta(\lambda - q) \, K_{cd}(q; \eta_c, \eta_d)
\]

\[
\gamma_{ecg}(\eta_c; -k, q, k - q) \gamma_{idl}(\eta_d; -k + q, -q, k)
\]

\[
G_{\lambda, ae}(k; \eta_a, \eta_c) G_{\lambda, gi}(|k - q|; \eta_c, \eta_d) G_{\lambda, lb}(k; \eta_d, \eta_b)
\]

**Kernel:**

\[
K_{\lambda, cd}(q; \eta_c, \eta_d) = G_{\lambda, cm}(q; \eta_c, 0) u_m P^0(q) u_n G^T_{\lambda, nd}(q; 0, \eta_d)
\]
further approximation: if $k \gg \lambda$ it can be integrated analytically!

\[
\frac{\partial}{\partial \lambda} G_{\lambda, ab}(k; \eta_a, \eta_b) = -\frac{1}{2} (e^{\eta_a} - e^{\eta_b})^2 \frac{k^2}{3} \int d^3 q \, \delta(\lambda - q) \frac{P^0(q)}{q^2} G_{\lambda, ab}(k; \eta_a, \eta_b)
\]

$G_{\lambda=0, ab}(k; \eta_a, \eta_b) = g_{ab}(\eta_a, \eta_b)$  \hspace{1cm} boundary condition

\[
G_{\lambda, ab}(k; \eta_a, \eta_b) = g_{ab}(\eta_a, \eta_b) \exp \left[ -\frac{k^2 \sigma_{\lambda}^2}{2} (e^{\eta_a} - e^{\eta_b})^2 \right]
\]

where \[
\sigma_{\lambda}^2 = \frac{1}{3} \int d^3 q \, \frac{P^0(q)}{q^2} \theta(\lambda - q)
\]
in perturbation theory, it corresponds to the summation of the infinite series of chain diagrams

here, it is obtained by a simple 1-loop integration!
A self-generated UV cutoff

Inserting this result in the expression for the RG kernel, we get:

$$K_{\lambda, cd}(q; \eta_c, \eta_d) = u_c u_d P^0(q) \exp \left[ -\frac{q^2 \sigma^2}{2} \left( (e^{\eta_c} - 1)^2 + (e^{\eta_d} - 1)^2 \right) \right]$$

The effect of modes with momenta larger than $$\sigma^{-1}_{\lambda} (e^\eta - 1)^{-1}$$ is exponentially screened.

The UV is much better behaved than one would guess from `usual' perturbation theory \((K_{\lambda, cd}(q; \eta_c, \eta_d) \rightarrow u_c u_d P^0(q)) \)!!
Confirmed by N-body simulations

smooth interpolation between the small $k$ (perturbative) and the large $k$ (resummed) results
(from Crocce and Scoccimarro, '06)
Conclusions

0) It is very important to quantify departures from linear theory in order to compare cosmological models with future galaxy surveys. The $0<z<1$ range is the most delicate one for DE studies;

1) The compact perturbation theory formulated by Crocce and Scoccimarro is a very convenient starting point for applying RG techniques to cosmology;

2) Exact RG equations can be derived for any kind of correlation function and for the scale-dependent growth factor;

3) Systematic approximation schemes, based on truncations of the full hierarchy of equations, can be applied, borrowing the experience from field theory;

4) A simple approximation scheme already shows the emergence of an intrinsic UV cutoff in the RG running;

5) Immediate lines of development include: computation of the power spectrum and of the bispectrum, improved approximations for the propagator, redshift-space distortions, non-gaussian initial conditions.